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Weapon accuracy assessment for elliptical normal miss distances is accomplished through the use of statistical tolerance circles. While tolerance circles for circular normal miss distances have been available in the literature for some time, such circles for the elliptical normal case have not been available heretofore. They are formulated for the bivariate normal distributions with unequal variances. The procedure is approximate in that it provides circles which contain at least 100P% of

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(gamma)

the population with approximately (vice exactly) 100γ% confidence. Its closeness is examined in detail and is shown to provide confidence which is very close to 100γ%. 100(gamma)%

The procedure is extended to the uncorrelated k dimensional normal distribution but the closeness is not studied for $k > 2$. Its application is general and is not restricted to miss distance data.

FOREWORD

The work covered in this technical report was performed in the Mathematical Statistics Branch, Operations Research Division, Strategic Systems Department. The date of completion was January, 1978.

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I. INTRODUCTION

The assessment of delivery accuracy for a weapon is usually initiated through test firings. The results of these test firings are miss distances of the rounds from the target center in the x (cross range) and y (down range) directions, usually denoted by $\{x_i, y_i\}_{i=1}^{n}$. These n pairs of miss distances are then used to compute an estimate of a parameter or parameters which describes the accuracy of the weapon. The parameter most commonly estimated is the CEP (Circular Error Probable), the radius of a circle within which 50% of the future rounds will fall. This estimate provides a single number which is customarily used to describe the accuracy of the weapon under study.

The procedure described above can be improved by incorporating the concept of statistical tolerance limits. This topic is addressed in Reference [8] but will be briefly restated here for completeness. Consider the case where the miss distances from the target center follow a circular normal distribution. This distribution is given by

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/(2\sigma^2)} \qquad (1)$$

where σ is the common standard deviation of the miss distances (common to both the x and y directions). Under this distributional assumption, the relationship between CEP and σ is well known to be

$$CEP = 1.1774\,\sigma \,. \qquad (2)$$

If one knew the value of the population parameter σ, then CEP above, defined in terms of σ, provides the radius of a circle which contains exactly 50% of the bivariate probability and, hence, will contain 50% of the future rounds. However, σ is usually not known and must be estimated from the test data. One such estimate based on the theory of maximum likelihood is

$$\hat{\sigma} = \left[\sum_{i=1}^{n} (x_i^2 + y_i^2)/2n\right]^{\frac{1}{2}} \qquad (3)$$

where the circumflex (or hat) above σ is used to signify an estimate of the true population parameter σ. Hence, an estimate of CEP is obtained by replacing σ in (2) with $\hat{\sigma}$ which yields

$$\hat{CEP} = 1.1774\ \hat{\sigma}\ . \tag{4}$$

(This estimate is slightly biased but it is rarely corrected since the bias is small and the correction factor is complicated.) $\hat{CEP}$ above is only a point estimate of the true CEP, and, as such, it yields no information regarding the probability that the circle it defines contains 50% or more of the population. This probability has been evaluated in Reference [8] for the circular case under discussion. The results show that it does not exceed .50 for any finite sample size. This means that one has at most 50% confidence that a circle of radius $\hat{CEP}$ will encompass 50% or more of the future impact points from the weapon under test.

To increase this probability or confidence to a more acceptable level, it is clear that one must increase the multiplying factor in (4) above the customary 1.1774. Tables of these multiplying factors are provided in Reference [8] for the case of circular normally distributed miss distances. These factors enable the analyst to construct circles which encompass at least 100P% of the population (future rounds) with 100γ% confidence for a wide variety of realistic values of n, P, and γ. Circles of this type are referred to in statistical literature as tolerance circles, tolerance regions or tolerance limits. In general, (P,γ) tolerance limits are functions of the sample data and define a region which contains at least 100P% of the population with 100γ% confidence.

On the basis of the above, one can utilize developed procedures to construct tolerance circles provided the miss distances follow a circular normal distribution. He can use the multiplying factor which will provide any level of prescribed confidence not only for 50% of the population but for any desired percentage. The problem here is that miss distance distributions are not always circular normal. In fact, more often than not, they tend to follow an elliptical distribution. The purpose of this report is to provide a procedure for constructing tolerance circles for the case where the miss distance distribution follows an uncorrelated bivariate normal distribution with unequal variances (hereinafter referred to as elliptical normal). The procedure to be developed in the ensuing sections is not exact in the sense of providing circles

which contain at least 100P% of the population with exactly 100γ% confidence. However, the confidence afforded by these approximate tolerance circles is very close to 100γ%. The closeness has been examined through a computer simulation, the results of which are provided in a later section.

II. TOLERANCE CIRCLES: σ_X and σ_Y KNOWN

It would be instructive at this point to examine the problem of finding the radius of the 100P% circle when the miss distances are elliptical normal and the population parameters are assumed known. Such circles can be thought of as tolerance circles when the sample size n is infinite and the confidence coefficient $\gamma = 1.00$. This development will pave the way and ease the burden in the development of tolerance circles for the more practical case where σ_X and σ_Y are assumed unknown.

We shall first briefly review the circular case. Let the bivariate random variable (X,Y) represent the miss distance of a round in the x and y directions, respectively. If this miss distance follows a circular normal distribution, the probability density of (X,Y) is given by equation (1). From equation (1), it is easy to show that the probability density of the radial miss distance $R = (X^2 + Y^2)^{\frac{1}{2}}$ is

$$g_C(r) = (r/\sigma^2)\, e^{-r^2/2\sigma^2}, \quad r > 0 \tag{5}$$

where the subscript C denotes circular. Furthermore, one can obtain the cumulative distribution function of R in closed form, i.e.,

$$G_C(r) = \mathrm{Prob}(R \le r) = \int_0^r g_C(t)\, dt = 1 - e^{-r^2/2\sigma^2} . \tag{6}$$

To obtain the well known relation between σ and CEP, one merely finds the value of r for which $G_C(r) = .5$. Solving

$$.5 = 1 - e^{-r^2/2\sigma^2} ,$$

one finds r = CEP = 1.1774σ which was given in equation (2). To find the radius of the general 100P% circle, C_p, one solves

$$P = 1 - e^{-r^2/2\sigma^2}$$

for r which yields

$$r = C_p = \sigma \{-2 \ln (1-P)\}^{\frac{1}{2}}. \tag{7}$$

The development is quite easy for the circular case since only one population parameter, σ, is involved and the cumulative distribution function of R can be expressed in closed form. Let us now turn to the elliptical case.

If the miss distance follows an elliptical normal distribution, the density of (X,Y) is given by

$$f(x,y) = (1/2\pi\sigma_X\sigma_Y)\, e^{-[(x/\sigma_X)^2 + (y/\sigma_Y)^2]/2} \tag{8}$$

where σ_X and σ_Y are the miss distance standard deviations in the x and y directions, respectively. The probability density of the radial miss distance R is more complicated in this case and is given by

$$g_E(r) = (r/\sigma_X\sigma_Y)\, e^{-ar^2}\, I_0(br^2), \quad r \geq 0 \tag{9}$$

where the subscript E denotes elliptical and where

$$a = \frac{\sigma_Y^2 + \sigma_X^2}{(2\sigma_X\sigma_Y)^2}, \qquad b = \frac{\sigma_Y^2 - \sigma_X^2}{(2\sigma_X\sigma_Y)^2}$$

and $I_0(x)$ is the Modified Bessel Function of the First Kind and Zero Order, i.e.,

$$I_0(x) = \frac{1}{\pi} \int_0^{\pi} e^{-x \cos \theta} \, d\theta \, .$$

The derivation of the density of R in (9) is obtained by applying the usual transformation from Cartesian to polar coordinates to (8). The complete derivation is provided by Chew and Boyce in Reference [1].

The cumulative distribution function of R is defined by

$$G_E(r) = \mathrm{Prob}(R \le r) = \int_0^r g_E(t) \, dt. \tag{10}$$

However, it is clear that $G_E(r)$ cannot be expressed in closed form since the integrand contains an integral which cannot be expressed in closed form. Hence, the radius of the 100P% circle for the elliptical case cannot be expressed by a simple formula as it was for the circular case (equation (7)). One must solve

$$P = G_E(r) \tag{11}$$

for $r = C_P$ which can only be accomplished by numerical methods.

The direct solution of equation (11) (solution of P for fixed values of r, σ_X, and σ_Y) can be obtained by using an efficient algorithm developed by DiDonato and Jarnagin which is documented in Reference [3]. The inverse solution (solution of r for fixed values of P, σ_X, σ_Y) can be obtained from tables prepared by DiDonato and Jarnagin and set forth in the same document. The point to be made is that to find the radius of the 100P% circle for the elliptical case, one must resort to tables or engage in extensive numerical efforts.

Thus far, no mention has been made about the CEP for the elliptical case. The reason is that the CEP per se is associated only with the circular case. However, there is certainly a circle centered at the target center which contains 50% of the bivariate probability. The radius of the circle is just a special case of C_P

above with P = .50. To distinguish it from the CEP for the circular case, the radius of the 50% circle for the elliptical case is sometimes referred to as an "equivalent CEP" (ECEP). The reason for distinguishing between the two is to avoid possible misuse of the CEP for the elliptical case. For example, if one were given a CEP with no mention that it pertained to the elliptical case, he may be tempted to divide it by 1.1774 and use the resulting σ in equation (6) to obtain "hit probabilities." Such probabilities are erroneous for values of $r \neq$ ECEP unless the population distribution is circular normal.

To avoid a table "look up" for the ECEP, several approximations have become fashionable over the years. The three most common are given below:

$$\mathrm{ECEP} \approx 1.1774\ (\sigma_X\ \sigma_Y)^{\frac{1}{2}} \tag{12}$$

$$\mathrm{ECEP} \approx 1.1774\ [\ (\sigma_X + \sigma_Y)/2] \tag{13}$$

$$\mathrm{ECEP} \approx 1.1774\ [\ (\sigma_X^2 + \sigma_Y^2)/2]^{\frac{1}{2}}\ . \tag{14}$$

These approximations are discussed by Groves in Reference [4] where it is shown that unless σ_X is very close to σ_Y only (13) has merit and then only if the ratio of the larger to the smaller of these two standard deviations is less than about five. A fourth approximation will be given later in this report as a limiting form of the expression to be derived for tolerance circles. This fourth approximation is not restricted to the radius of the 50% circle but has general application to the 100P% circle. It is found to be more accurate than any of the above three for approximating the radius of the 50% circle.

Throughout this section, it has been assumed that σ_X and σ_Y are known a priori. This is rarely the case in that σ_X and σ_Y are usually estimated from test data. One could use these estimates in place of σ_X and σ_Y in equation (11) to obtain point estimate of the ECEP or the more general C_P. However, point estimation for the elliptical case yields no more information than for the circular case. Therefore, let us turn our attention to the problem of deriving tolerance limits for the elliptical case when the parameters are estimated from test data.

III. APPROXIMATE TOLERANCE CIRCLES: σ_X AND σ_Y UNKNOWN

Suppose n rounds are fired at a target and yield miss distances $\{x_i, y_i\}_{i=1}^{n}$. Under the assumption that the miss distances follow an elliptical normal distribution, the probability density for the population of miss distances is given by equation (8). The standard deviations σ_X and σ_Y are unknown, but their maximum likelihood estimates are

$$\hat{\sigma}_X = \{\sum_{i=1}^{n} x_i^2/n\}^{\frac{1}{2}} \qquad \hat{\sigma}_Y = \{\sum_{i=1}^{n} y_i^2/n\}^{\frac{1}{2}} .$$

To construct a tolerance circle in accordance with the definition on page 3, we need to find a function of the sample data, say $L(x_1, \ldots, x_n, y_1, \ldots, y_n)$ which satisfies

$$\text{Prob}\left\{ \iint_{x^2+y^2 \le L^2} \frac{1}{2\pi\, \sigma_X\, \sigma_Y}\, e^{-[(x/\sigma_X)^2 + (y/\sigma_Y)^2]/2}\, dxdy \ge P \right\} = \gamma \quad (15)$$

where the arguments of L have been deleted. In the above, L is the radius of a circle which contains at least 100P% of the population with 100γ% confidence. It is easy to show that $\hat{\sigma}_X$ and $\hat{\sigma}_Y$ above are sufficient for their respective parameters. Hence, the radius L of the tolerance circle can be expressed as a function of these two estimates vice all 2n observations. That is, we can express L in functional form as $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ vice $L(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Equation (15) is a general expression of the problem. However, the problem can be expressed more simply since a tolerance circle for this case is equivalent to an upper tolerance bound for the distribution of the radial error R as given by equation (9). Hence, the problem can be restated as one of finding $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ which satisfies

$$\mathrm{Prob}\left\{\int_0^{L(\hat\sigma_X,\hat\sigma_Y)} (r/\sigma_X\sigma_Y)\ e^{-ar^2}\ I_0(br^2)\ dr \geq P\right\} = \gamma \qquad (16)$$

where a and b are the functions of σ_X and σ_Y defined earlier. The function $L(\hat\sigma_X, \hat\sigma_Y)$ above could be found if (a) the integral in (16) could be expressed in closed form (as in the circular case) or (b) a suitable transformation could be found to express the integrand free of the unknown population parameters σ_X and σ_Y. Condition (a) clearly does not hold in this case, and with respect to condition (b), a suitable transformation has not been found. Hence, an approximate solution will be sought.

The integrand in (16) is the distribution of the radial miss distance $R = (X^2 + Y^2)^{\frac{1}{2}}$ under the assumption of elliptical normality on the bivariate random variable (X,Y). Consider now an approximation to the distribution of a function of R. Under our distributional assumption, $X \sim N(0, \sigma_X^2)$, $Y \sim N(0, \sigma_Y^2)$, and X and Y are independent where "~" signifies "is distributed according to" and $N(\mu,\sigma^2)$ signifies a normal distribution with mean μ and variance σ^2. It is well known that the square of a standard normal variable (N(0,1)) is a chi-square variable and that independent chi-square variables are additive where the density of a chi-square variable with ν degrees of freedom (χ^2_ν) is given by

$$h(w) = \frac{1}{\Gamma(\nu/2)\ 2^{\nu/2}}\ w^{(\nu/2)-1}\ e^{-w/2},\ w > 0\ . \qquad (17)$$

Since X and Y are not standard normal variables, $R^2 = X^2 + Y^2$ is not a chi-square variable. However, the distribution of

$$U = \frac{\nu R^2}{\sigma_X^2 + \sigma_Y^2} \qquad (18)$$

can be approximated with a chi-square density with ν degrees of freedom where

$$\nu = \frac{(\sigma_X^2 + \sigma_Y^2)^2}{\sigma_X^4 + \sigma_Y^4}\ . \qquad (19)$$

This approximation is a special case of a more general approximation due to Satterthwaite (see Reference [6]). The general form is as follows:

$$\text{Let } \nu_i X_i^2/\sigma_i^2 \sim \chi^2_{\nu_i}, \quad i = 1, \ldots, k$$

and let $X_1, X_2, \ldots, X_k$ be independent.

$$\text{Then } U = \frac{\nu \sum_{i=1}^{k} a_i X_i^2}{\sum_{i=1}^{k} a_i \sigma_i^2} \sim \chi^2_\nu \tag{20}$$

approximately where the a_i are constant and

$$\nu = \frac{\left(\sum_{i=1}^{k} a_i \sigma_i^2 \right)^2}{\sum_{i=1}^{k} \left((a_i^2 \sigma_i^4/\nu_i) \right)} .$$

For our application above, $k = 2$, $a_1 = a_2 = 1$, and $\nu_1 = \nu_2 = 1$.

The approximate distribution of U in (18) could be used to find the approximate distribution of R. However, one can restate the problem in terms of U via

$$\text{Prob}\{\int_0^A h(u)du \geq P\} \approx \gamma \tag{21}$$

where $A = \nu L^2/(\sigma_X^2 + \sigma_Y^2)$ and $h(u)$ is the chi-square density given in (17) with degrees of freedom ν given by (19). The inequality in (21) holds if and only if $A \geq \chi^2_{\nu,P}$ where $\chi^2_{\nu,P}$ is the 100P% point for a chi-square variable with ν degrees of freedom. That is, if $U \sim \chi^2_\nu$, then $\chi^2_{\nu,P}$ is such that

$$\text{Prob}\ (U < \chi^2_{\nu,P}) = P\ .$$

Hence, (21) can be restated as

$$\text{Prob}\left\{\frac{\nu L^2}{\sigma_X^2 + \sigma_Y^2} \geq \chi^2_{\nu,P}\right\} \approx \gamma \tag{22}$$

or equivalently,

$$\text{Prob}\left\{\sigma_X^2 + \sigma_Y^2 \leq \frac{\nu L^2}{\chi^2_{\nu,P}}\right\} \approx \gamma\ . \tag{23}$$

To complete the solution, we need to obtain the upper 100γ% confidence bound for $\sigma_X^2 + \sigma_Y^2$. We then equate this bound to $\nu L^2/\chi^2_{\nu,P}$ and solve for L.

To obtain a confidence bound for $\sigma_X^2 + \sigma_Y^2$, we again resort to Satterthwaite's approximation (20). Under the initial normality assumption,

$$n\hat{\sigma}_X^2/\sigma_X^2 \sim \chi_n^2 \text{ and } n\hat{\sigma}_Y^2/\sigma_Y^2 \sim \chi_n^2\ .$$

Using (20) with $k = 2$, $a_1 = a_2 = 1$, and $\nu_1 = \nu_2 = n$, the distribution of

$$\frac{\nu'(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2)}{\sigma_X^2 + \sigma_Y^2}$$

can be approximated with a chi-square distribution with ν' degrees of freedom where

$$\nu' = \frac{(\sigma_X^2 + \sigma_Y^2)^2}{\sigma_X^4/n + \sigma_Y^4/n} = n\nu \ .$$

Using this approximation, the upper $100\gamma\%$ confidence bound for $\sigma_X^2 + \sigma_Y^2$ is

$$\frac{n\nu(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2)}{\chi^2_{n\nu,1-\gamma}} \ .$$

We now set

$$\frac{\nu L^2}{\chi^2_{\nu,p}} = \frac{n\nu(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2)}{\chi^2_{n\nu,1-\gamma}}$$

and solve for L. We find

$$L\,(\hat{\sigma}_X,\,\hat{\sigma}_Y) = \left\{ \frac{n\chi^2_{\nu,P}}{\chi^2_{n\nu,1-\gamma}} \right\}^{\frac{1}{2}} (\hat{\sigma}_X^2 + \hat{\sigma}_Y^2)^{\frac{1}{2}} \; . \tag{24}$$

The parameter ν in (24) is a function of the parameters σ_X^2 and σ_Y^2 which are unknown. Hence, ν must be estimated from the data by replacing σ_X^2 with $\hat{\sigma}_X^2$ and σ_Y^2 with $\hat{\sigma}_Y^2$ in formula (19) for ν. One notes that ν is not restricted to the integers in (24) so that to apply this formula one needs a table of chi-square percentage points for fractional degrees of freedom. A table of this type has recently been prepared by DiDonato and Hageman (Reference [2]) and will be utilized in the example which follows.

Suppose 15 rounds are fired at a target and provide the miss distances shown in Table 1. Estimates of σ_X and σ_Y obtained from the data are $\hat{\sigma}_X = 85.11$ and $\hat{\sigma}_Y = 20.55$. Let us use these values to construct (P,γ) tolerance circles for $P = .50$, $\gamma = .90$ (the circle which contains 50% of the population of future rounds with 90% confidence) and for $P = .50$, $\gamma = .95$ (the circle which contains 50% of the population of future rounds with 95% confidence). For both circles, the value of

$$(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2)^{\frac{1}{2}} = [(85.11)^2 + (20.55)^2]^{\frac{1}{2}} = 87.56.$$

we now need to calculate ν which, according to formula (19) is

$$\nu = \frac{[(85.11)^2 + (20.55)^2]^2}{(85.11)^4 + (20.55)^4} = 1.116 \; .$$

and $n\nu = 15\nu = 16.74$. The value of the 50% (value of P) point is needed for a chi-square with 1.116 degrees of freedom as well as values of the 5% and 10% (values of $1-\gamma$) points for a chi-square with 16.74 degrees of freedom. Reference [2] cited above provides tabular values of chi-square percentage points for $\nu = .1(.1)13(.12)26(.25)42.5(.5)512$. The points needed for our degrees of freedom can be obtained using these tables with linear interpolation, or they can be obtained exactly by using the program documented in the report. Linear interpolation of tabular values provides

TABLE 1

Hypothetical Miss Distances For Sample Problem

x	y
11.71	- 3.40
51.94	-19.68
- 87.60	-20.80
31.11	12.35
-104.12	15.68
23.95	1.53
- 15.10	16.60
96.24	-12.55
157.33	-47.80
- 74.22	3.43
68.04	-14.78
- 90.03	-24.70
-110.75	8.05
95.01	-21.05
111.74	32.15

$$\hat{\sigma}_X = \left\{ \sum_{i=1}^{15} x_i^2/15 \right\}^{\frac{1}{2}} \qquad \hat{\sigma}_Y = \left\{ \sum_{i=1}^{15} y_i^2/15 \right\}^{\frac{1}{2}}$$

$$= 85.11 \qquad = 20.55$$

$$\chi^2_{1.116,\ .50} = .5564$$

$$\chi^2_{16.74,\ .10} = 9.8836$$

$$\chi^2_{16.74,\ .05} = 8.4863\ .$$

The exact values, obtained by running the program, are .5564, 9.8822, and 8.4844, respectively. The values obtained by linear interpolation are the values which would usually be available to the analyst. They are sufficiently accurate and will be used to complete our sample problem.

For $(P,\nu) = (.50,\ .90)$, we find

$$L(\hat{\sigma}_X,\ \hat{\sigma}_Y) = \left\{ \frac{(15)(.5564)}{9.8836} \right\}^{\frac{1}{2}} (87.56) = 80.46,$$

and for $(P,\nu) = (.50,\ .95)$, we find

$$L(\hat{\sigma}_X,\ \hat{\sigma}_Y) = \left\{ \frac{(15)(.5564)}{8.4863} \right\}^{\frac{1}{2}} (87.56) = 86.83\ .$$

The point estimate of the ECEP for this case is 61.42. It is obtained by entering the tables in Reference [3] with $c = 20.55/85.11 = .2415$ and $P = .50$ to obtain .7217 which when multiplied by $\hat{\sigma}_X$ yields the above result. It has an associated confidence of less than 50%, i.e., we are less than 50% confident that a circle of radius 61.42 contains at least 50% of the population of impacts.

The data points for this example and all three circles are shown in Figure 1. All three circles can be thought of as estimates of the "50% circle" but with different levels of confidence. One notes in this example and in general that the confidence increases as the circle radius increases.

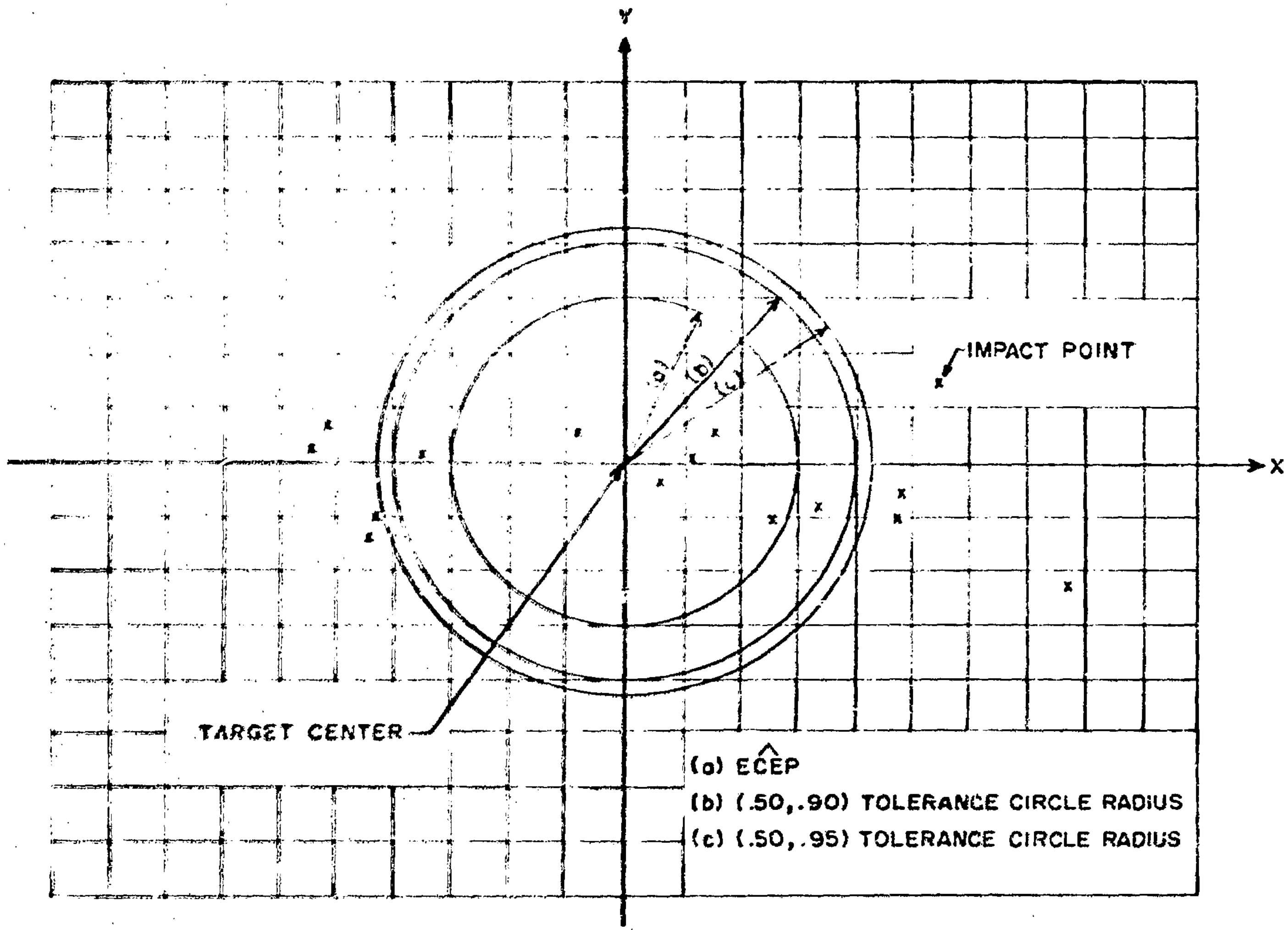
Y
X
IMPACT POINT
TARGET CENTER
(a)
(b)
(c)
(a) $\widehat{ECEP}$
(b) (.50,.90) TOLERANCE CIRCLE RADIUS
(c) (.50,.95) TOLERANCE CIRCLE RADIUS

FIGURE 1

IV. ACCURACY EVALUATION

Satterthwaite's approximation (20) was used twice during the formulation of the (P,ν) tolerance circle given by equation (24). Therefore, it is an approximate tolerance circle, and one should know the worth of this approximation in order to use it with "confidence." The purpose of this section is to evaluate the accuracy of the approximation; several approaches will be used in this endeavor. Let us first define $0 \le c = \sigma_Y/\sigma_X \le 1$. Then the first approach will be to examine $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ in (24) for the cases where $c = 1$ and $c = 0$. This will correspond, respectively, to the circular case (case where $\sigma_X = \sigma_Y$) and the univariate case (case where σ_X differs so much from σ_Y that the entire population is concentrated on a line). The second approach will be to examine the confidence afforded by the approximation for small sample sizes n using a Monte Carlo simulation. The last approach will be to examine the limit of $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ as n increases without bound. The results of these examinations should be sufficient to pass judgment on the worth of equation (24).

If $c = 1$, then $\sigma_X = \sigma_Y = \sigma$, say, and $\hat{\sigma}_X^2$ and $\hat{\sigma}_Y^2$ in (24) are both estimating the common variance σ^2 so that $\hat{\sigma}_X^2 + \hat{\sigma}_Y^2$ becomes $2\hat{\sigma}^2$. Also, ν, in equation (19) becomes

$$\nu = \frac{(\sigma^2 + \sigma^2)^2}{\sigma^4 + \sigma^4} = \frac{4\sigma^4}{2\sigma^4} = 2 .$$

Hence, if $c = 1$, $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ becomes

$$L(\hat{\sigma}) = \left\{\frac{n\,\chi^2_{2,P}}{\chi^2_{2n,1-\gamma}}\right\}^{\frac{1}{2}} (2\hat{\sigma}^2)^{\frac{1}{2}}$$

$$= \left\{\frac{2n\,\chi^2_{2,P}}{\chi^2_{2n,1-\gamma}}\right\}^{\frac{1}{2}} \hat{\sigma} . \qquad (25)$$

Equation (25) coincides exactly with the exact solution for tolerance circles for the circular case provided by Thomas and Crigler in Reference [7]. (Reference [7] is a refinement of the work in Reference [8] for publication in the open literature.) Therefore, if $\sigma_X = \sigma_Y$, the approximate solution in (24) corresponds to the exact solution for the circular case.

To examine c = 0, we shall simply set $\sigma_Y = 0$. Here we find $\hat{\sigma}_X^2 + \hat{\sigma}_Y^2 = \hat{\sigma}_X^2$ and ν in equation (19) equals one so that $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ becomes

$$L(\hat{\sigma}_X) = \left\{ \frac{n \; \chi^2_{1,P}}{\chi^2_{n,1-\gamma}} \right\}^{\frac{1}{2}} \hat{\sigma}_X \; . \tag{26}$$

Equation (26) is the exact expression for the upper tolerance bound for a univariate normal distribution when the mean is known a priori to be zero. (See the development of the univariate case by Hald, for example, in Reference [5].) Therefore, the approximate solution in (24) becomes the exact solution in the degenerate case where one of the standard deviations is zero.

The results of the last two paragraphs indicate that the approximation provides what is needed for values of c close to zero or one. For mid-range values of c, we shall have to resort to Monte Carlo simulation. A single replicate of the simulation process used in this study is described below:

(i) For given values of P, γ, σ_X, and σ_Y, a sample of size n was generated from an elliptical normal distribution using Monte Carlo sampling techniques.

(ii) Using the data points generated in (i), $\hat{\sigma}_X$, $\hat{\sigma}_Y$, and ν were computed.

(iii) Using the value of ν computed in (ii) and the specified values of P and γ, the required fractional chi-square percentage points were computed and $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ formed.

(iv) The integral I of the elliptical normal distribution with parameters σ_X and σ_Y was computed over a circle of radius $L(\hat{\sigma}_X, \hat{\sigma}_Y)$.

The above process was repeated N times, and the confidence coefficient was estimated by the ratio of the number of replicates in which $I \geq P$ to the total number of replicates N. If the equation for L were exact, then γ would lie within sampling variation of $\hat{\gamma}$ (the estimate of the true confidence) for all values of P, γ, c, and n. Since L in (24) is an approximation, the departure of $\hat{\gamma}$ from γ (outside sampling variation) is a measure of the worth of the approximation.

The selected values of the input parameters P, γ, n, and c for the simulation were as follows:

P: .50, .90

γ: .90, .95

n: 5, 10, 20

c: 0, .05, .10, .20, .25, .33, .50, .57, .67, .80, 1.00.

Values of n were restricted to small samples in order to avoid excessive computer running time. The number of replicates N for each combination of parametric values was set at 10,000. This value insures (with 95% confidence) that the true confidence (not necessarily γ) lies within .01 of $\hat{\gamma}$. The results are shown in Figures 2-5 where $\hat{\gamma}$ is plotted as a function of c in multiple curves representing the three different values of n. Each figure represents a different (P,γ) combination. For each combination, horizontal lines have been drawn at $\gamma \pm .01$. Values of $\hat{\gamma}$ within the region enclosed by these lines can be considered within sampling variation of γ. Values outside the region represent values of $\hat{\gamma}$ outside sampling variation and indicate departures of the approximation from γ. One notes that in all four cases $\hat{\gamma}$ lies in the region of sampling variation for c close to zero. This merely confirms the results of the earlier discussions on the behavior of the approximation when σ_X and σ_Y are substantially different. One does not observe a similar behavior for all values of P, γ, and n for the case where c = 1. He does observe, however, that for the larger value of n(n = 20), $\hat{\gamma}$ lies either within or very close to the region of sampling variation. Departures also exist for the mid-range values of c (most notably for values of c close to .25) with the departure being less severe for the larger n than for the smaller one.

P = .5 γ = .90

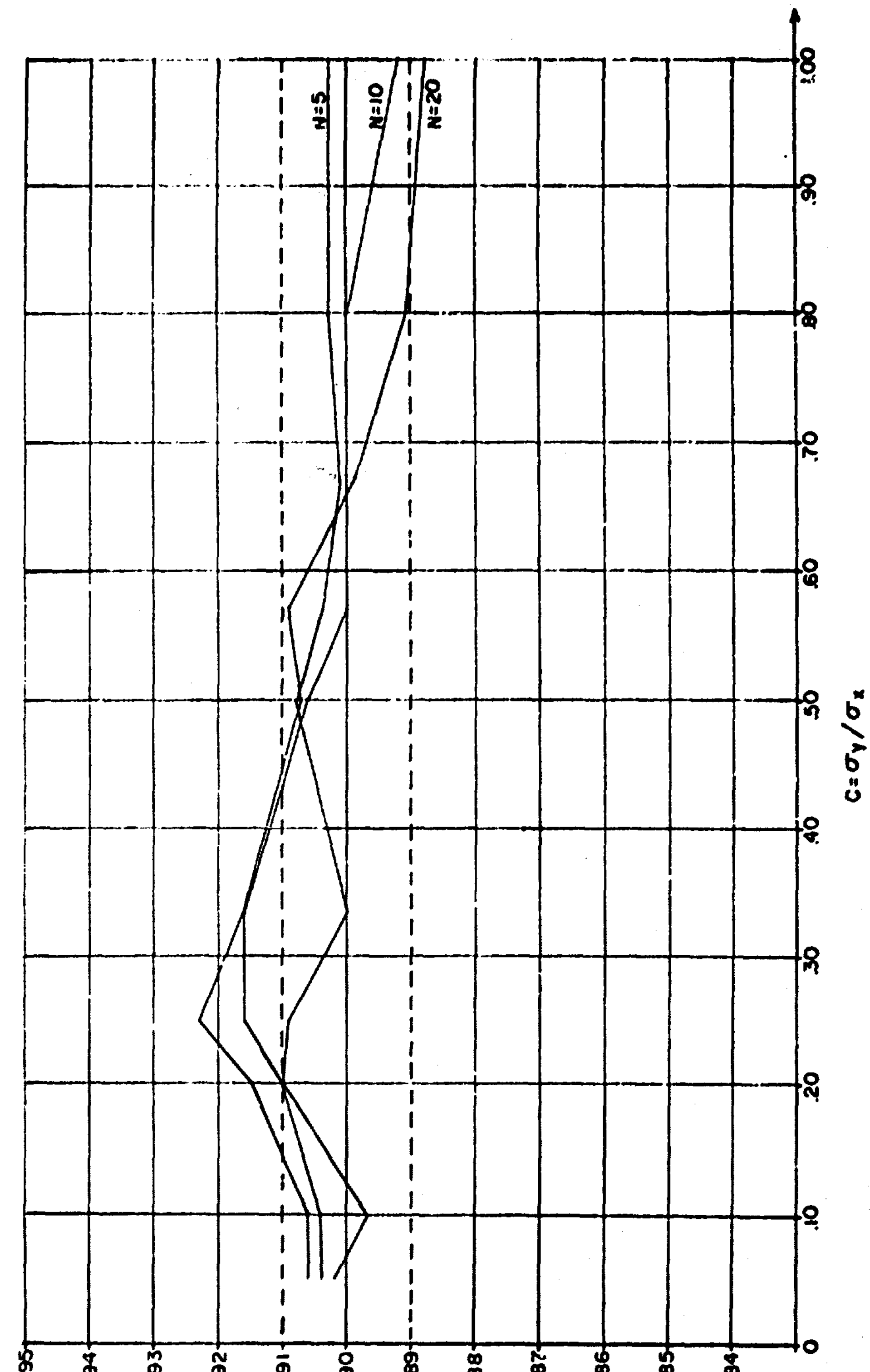

FIGURE 2

$\rho = .5 \quad \gamma = .95$

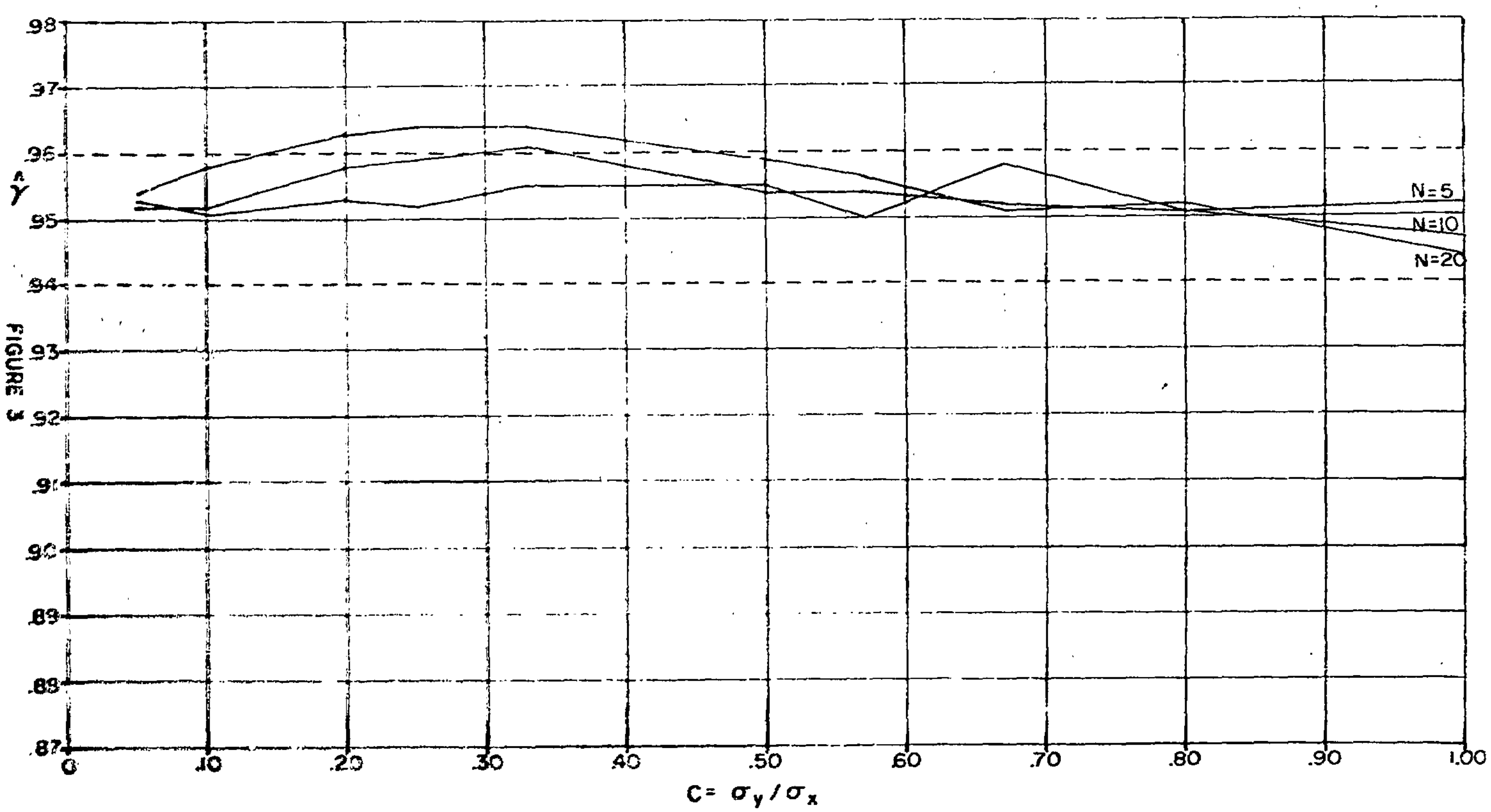

FIGURE 3

P = .90 γ = .90

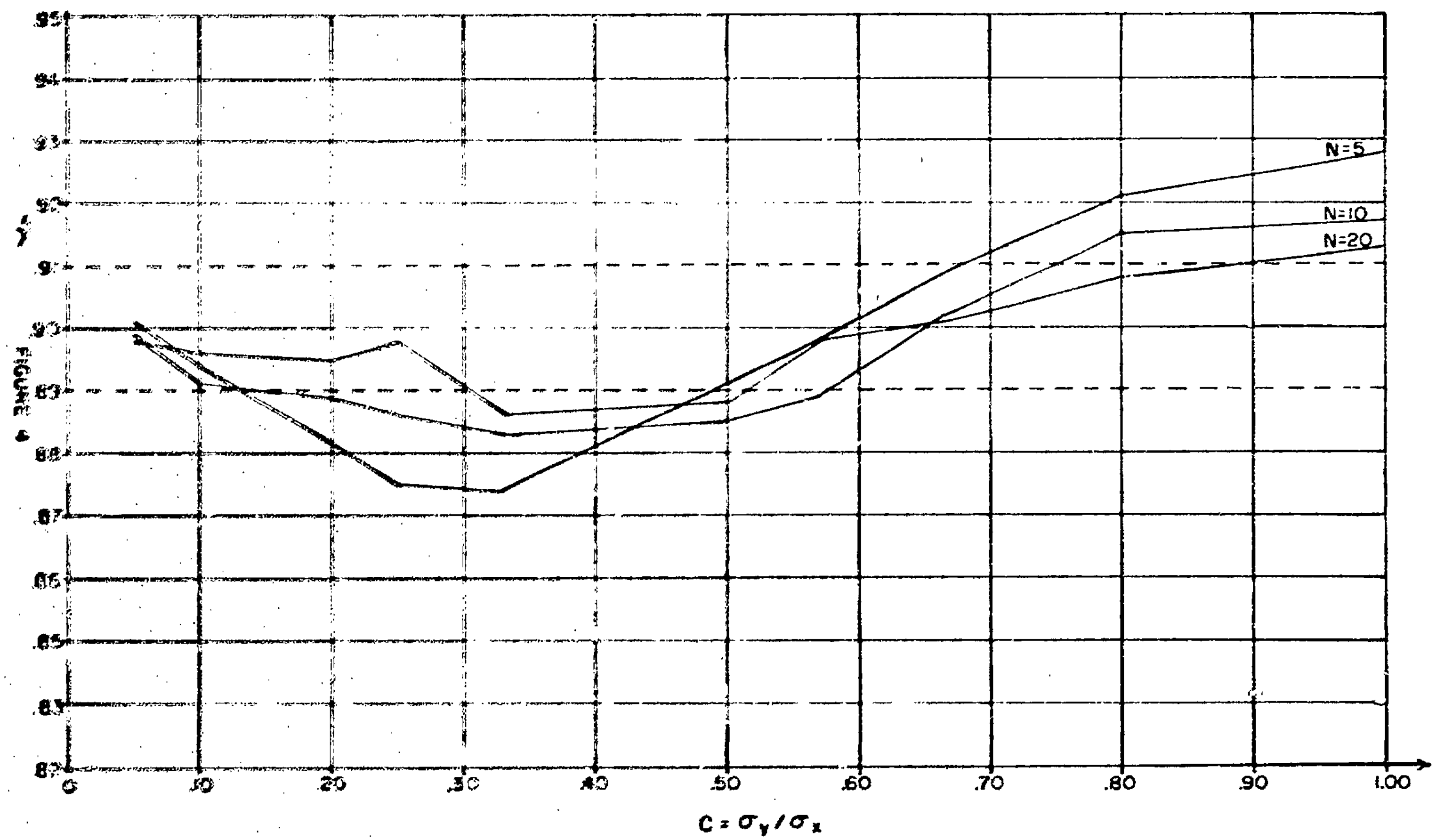

FIGURE 4

P = .9 γ = .95

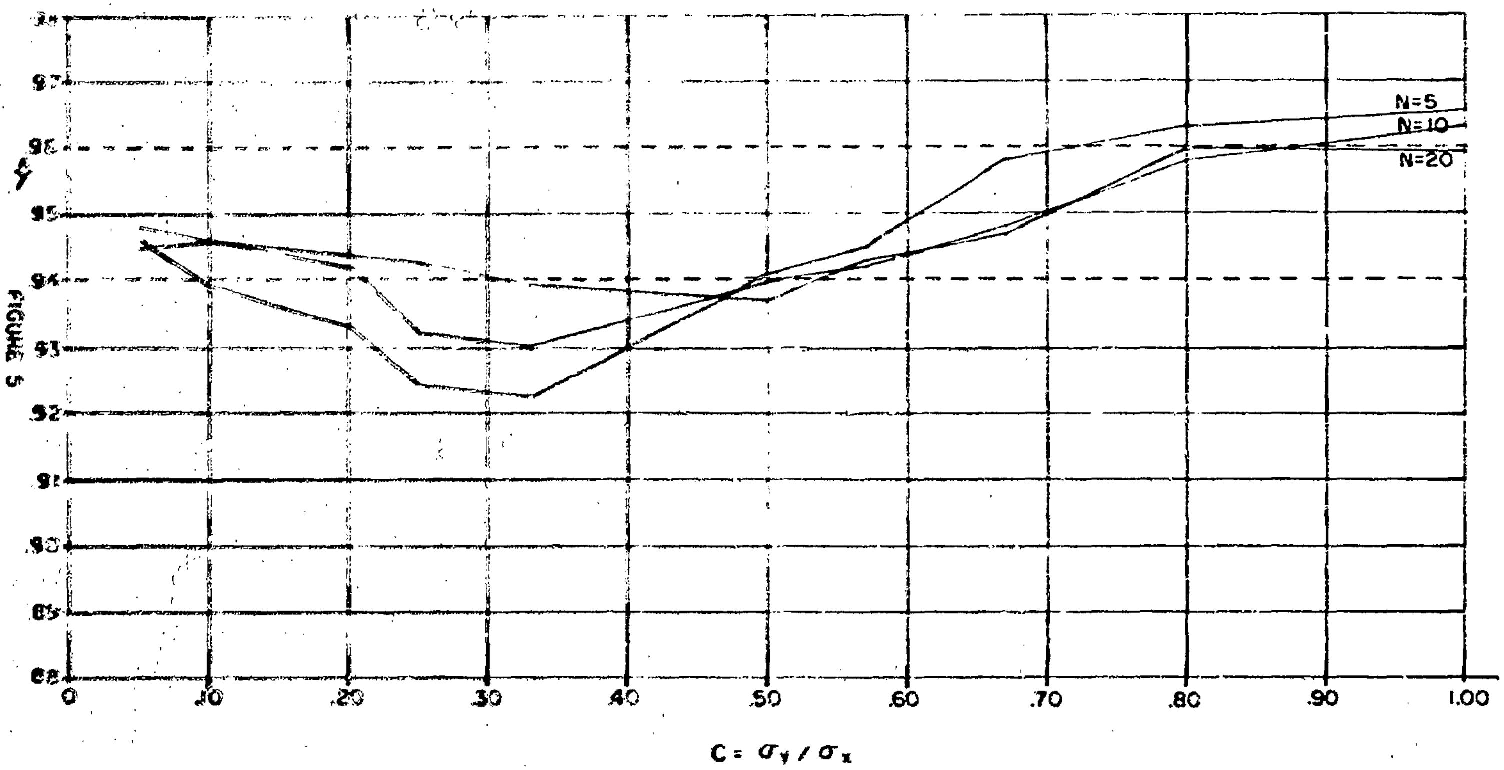

FIGURE 5

On the basis of Figures 2-5 and the above comments, one can conclude that the approximation is very good for values of c close to zero regardless of the values of P, γ, and n. For values of c close to one, the approximation is also very good if n is not too small. For other values of c, there are departures in the confidence afforded by the approximation which appear to increase as P increases and diminish as n increases. We now need to insure that the approximation continues to improve as n increases beyond the maximum value of 20 used in the simulation. Hence, we shall consider the limit of $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ as $n \to \infty$. It is easy to show that

$$\lim_{n \to \infty} L(\hat{\sigma}_X, \hat{\sigma}_Y) = \left\{ \frac{\chi^2_{\nu,P}}{\nu} \right\}^{\frac{1}{2}} (\sigma_X^2 + \sigma_Y^2)^{\frac{1}{2}} . \tag{27}$$

The right hand portion of the limiting expression is based on the fact that $\hat{\sigma}_X^2$ and $\hat{\sigma}_Y^2$ are consistent estimators and, hence, approach σ_X^2 and σ_Y^2, respectively, as $n \to \infty$. The left hand portion is based on the fact that as the degrees of freedom $f \to \infty$, the distribution of χ_f^2/f is concentrated at unity. Hence, in the limit, all percentage points of a chi-square random variable divided by its degrees of freedom are unity. If approximation (24) continues to improve as $n \to \infty$, then the percent of the population encompassed by a circle with radius given by (27) should approach P. The percent of the population encompassed by circles of radius (27) was evaluated for c = 0.0(.1)1.0 and P = .50, .90, .95, and .99. The results are set out in Table 2 and show that while these percentages, designated P', are not exactly equal to P, they are very nearly equal to P, differing only in the 3rd decimal place. The meaning here is that the percent of the population encompassed by a circle of radius $L(\hat{\sigma}_X, \hat{\sigma}_Y) \to P'$ vice P as $n \to \infty$. This poses no real problem since P' is close to P. Hence, one can rest assured that for all practical purposes, $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ in (24) improves as n increases without bound.

We can summarize the worth of approximation (24) with the following remarks:

(a) It approaches the exact value as $c \to 0$ and appears to produce excellent results regardless of n, P, and γ for values of c close to zero.

TABLE 2

Evaluation of

$$p' = \iint\limits_{x^2+y^2<(\text{limit } L)^2} \frac{1}{2\pi\sigma_x\sigma_y} e^{-[(x/\sigma_x)^2+(y/\sigma_y)^2]/2}\, dxdy$$

c	p	p'	c	p	p'
.0	.50	.5000	.0	.90	.9000
.1	.50	.5011	.1	.90	.9004
.2	.50	.5032	.2	.90	.9014
.3	.50	.5018	.3	.90	.9028
.4	.50	.4981	.4	.90	.9042
.5	.50	.4962	.5	.90	.9052
.6	.50	.4964	.6	.90	.9049
.7	.50	.4976	.7	.90	.9034
.8	.50	.4989	.8	.90	.9017
.9	.50	.4997	.9	.90	.9004
1.0	.50	.5000	1.0	.90	.9000
.0	.95	.9500	.0	.99	.9900
.1	.95	.9500	.1	.99	.9899
.2	.95	.9501	.2	.99	.9897
.3	.95	.9503	.3	.99	.9893
.4	.95	.9506	.4	.99	.9890
.5	.95	.9510	.5	.99	.9889
.6	.95	.9513	.6	.99	.9890
.7	.95	.9511	.7	.99	.9893
.8	.95	.9508	.8	.99	.9897
.9	.95	.9502	.9	.99	.9899
1.0	.95	.9500	1.0	.99	.9900

(b) It approaches the exact value as $c \to 1$, but the results do depend on n.

(c) For intermediate values of c, there is some departure from the desired confidence, but even for a sample size as small as $n = 5$, it does not exceed 3% and is much less for most values of c.

(d) It improves for all values of the parameters as n increases.

Hence, the approximation appears to be suitable for general applications with little loss in accuracy even for sample sizes as small as $n = 5$.

V. MULTIDIMENSIONAL EXTENSION

Thus far our attention has been confined to the two dimensional case where miss distances are recorded in the (x,y) plane. This could be the ground plane or perhaps the plane normal to the trajectory at impact. In the case of an air burst weapon, however, the miss distances are three dimensional, and for other applications, the number of dimensions could exceed three. In these cases, we would be concerned with a k dimensional tolerance sphere vice a tolerance circle. Extending the work of the previous sections to tolerance k-spheres for the multidimensional case is straightforward and will be outlined below.

Let the multivariate random variable $(X_1, \ldots, X_k)$ represent the miss distance of a round in k directions. If this miss distance follows an uncorrelated multivariate normal distribution, its probability density is given by

$$f(x_1, \ldots, x_k) = \frac{1}{(2\pi)^{k/2}\,\sigma_1 \sigma_2 \cdots \sigma_k}\, e^{-\frac{1}{2}[(x_1/\sigma_1)^2 + \ldots + (x_k/\sigma_k)^2]} \quad (28)$$

where σ_i is the miss distance standard deviation in the ith direction. If a sample of n rounds is fired, the n miss distances $\{x_{1j}, \ldots, x_{kj}\}_{j=1}^{n}$ are used to estimate the σ_i. The method of maximum likelihood yields

$$\hat{\sigma}_i = \left\{ \sum_{j=1}^{n} x_{ij}^2/n \right\}^{\frac{1}{2}}, \quad i = 1, 2, \ldots, k.$$

To construct a tolerance k-sphere we need to find the radius L as a function of the $\hat{\sigma}_i$ such that

$$\text{Prob}\left\{ \int \cdots \int_S f(x_1, \ldots, x_k)\, dx_1 \ldots dx_k \geq P \right\} = \gamma \quad (29)$$

where the region of integration S is the interior region of the k-sphere $x_1^2 + \ldots + x_k^2 = L^2$. If we let $R^2 = X_1^2 + \ldots + X_k^2$, the application of (20) provides that

$$U = \frac{\nu R^2}{\sigma_1^2 + \ldots + \sigma_k^2} \sim \chi_\nu^2 \quad \text{approximately}$$

where

$$\nu = \frac{(\sigma_1^2 + \ldots + \sigma_k^2)^2}{\sigma_1^4 + \ldots + \sigma_k^4} \; . \tag{30}$$

One can now restate (29) in terms of the distribution of U via

$$\text{Prob}\left\{\int_0^A h(u)du \geq P\right\} \approx \gamma \tag{31}$$

where h(u) is the chi-square density with degrees of freedom ν given by (30) and $A = \nu L^2/(\sigma_1^2 + \ldots + \sigma_k^2)$. Equation (31) holds if and only if $A \geq \chi^2_{\nu,P}$. Hence, (31) can be rewritten in form

$$\text{Prob}\left\{\frac{\nu L^2}{\sigma_1^2 + \ldots + \sigma_k^2} \geq \chi^2_{\nu,P}\right\} \approx \gamma$$

or equivalently

$$\text{Prob}\left\{\sigma_1^2 + \ldots + \sigma_k^2 \leq \frac{\nu L^2}{\chi^2_{\nu,P}}\right\} \approx \gamma \; . \tag{32}$$

Applying (20) again, we find the upper $100\gamma\%$ confidence bound for $\sigma_1^2 + \dots + \sigma_k^2$ to be approximately

$$\frac{n\nu\,(\hat{\sigma}_1^2 + \dots + \hat{\sigma}_k^2)}{\chi^2_{n\nu,1-\gamma}} . \tag{33}$$

Equating (33) to the right hand side of the inequality in (32), one obtains

$$L(\hat{\sigma}_1, \dots, \hat{\sigma}_k) = \left\{ \frac{n\chi^2_{\nu,P}}{\chi^2_{n\nu,1-\gamma}} \right\}^{\frac{1}{2}} (\hat{\sigma}_1^2 + \dots + \hat{\sigma}_k^2)^{\frac{1}{2}} . \tag{34}$$

One notes that this general expression is very similar to the two dimensional expression given by (24). The difference lies in the formula for ν and the additional terms in the right hand portion to account for the increased dimensionality. This expression for L is the radius of a k dimensional sphere which contains at least 100P% of the population with confidence approximately equal to $100\gamma\%$. The worth of this approximation has not been studied for dimensionality above two.

VI. ANOTHER APPROXIMATION: PARAMETERS KNOWN

It was shown in Section II that when the parameters are known a priori, the ECEP or the radius of the general 100P% circle for the elliptical case can be obtained exactly but only by table look-up or extensive calculations. Several approximations were shown ((12), (13), and (14)) which are oftentimes used for ECEP calculations to avoid the exact procedure. A fourth approximation which can be applied to the general case of the 100P% circle is simply the limiting form of $L(\hat{\sigma}_X, \hat{\sigma}_Y)$ in (24) as $n \to \infty$. This form was given in equation (27) in Section IV and upon minor rearrangement becomes

$$C_P \approx \left\{ 2\chi^2_{\nu,P}/\nu \right\}^{\frac{1}{2}} \left[(\sigma_X^2 + \sigma_Y^2)/2 \right]^{\frac{1}{2}} \qquad (35)$$

where ν is given by (19).

One notes from Table 2 that this approximation is exact for the degenerate cases where $c = \sigma_Y/\sigma_X = 0, 1$. Also from Table 2, we see that for midrange values of c, a circle of radius (35) contains very nearly 100P% of the population, differing from P only in the 3rd decimal place. However, to employ approximation (35), we need values of $\chi^2_{\nu,P}$ for fractional ν. To obtain such values, we have to resort to tables or engage in heavy calculations. These are the same alternatives we faced for the exact solution. Therefore, one could argue that if the approximation involves the use of tables of percentage points for the chi-square with fractional degrees of freedom (Reference [2]), why not use inverse tables of the circular coverage function (Reference [3]) to obtain the exact solution? This is a valid argument, and no attempt will be made to refute it except to say that many analysts feel more comfortable using a multiplying factor for some function of σ_X and σ_Y than they do using the table look-up for the exact solution. Furthermore, if one confines his attention to the ECEP, he can readily construct a short table of such factors for easy access. Table 3 below is a short table of this type where the percentage points have been taken from Reference [2].

To illustrate the use of Table 3 and to compare approximation (35) with the exact solution and the other approximations, consider the following example: If $\sigma_Y = 15$ ft. and $\sigma_X = 30$ ft.,

TABLE 3

Multiplying Factors of $[(\sigma_X^2 + \sigma_Y^2)/2]^{\frac{1}{2}}$ for ECEP Approximation

ν	$[2\,\chi^2_{\nu,.50}/\nu]^{\frac{1}{2}}$
1.0	.9538
1.1	.9928
1.2	1.0258
1.3	1.0542
1.4	1.0789
1.5	1.1005
1.6	1.1195
1.7	1.1365
1.8	1.1516
1.9	1.1652
2.0	1.1774

$$\nu = \frac{(\sigma_X^2 + \sigma_Y^2)^2}{\sigma_X^4 + \sigma_Y^4}$$

find the ECEP by the exact method and compare with the four approximate solutions. We obtain the exact solution from tables in Reference [3]. The first three approximations are straightforward and require no elaboration. The last approximation, (35), involves the use of Table 3. For this case

$$\nu = \frac{[(30)^2 + (15)^2]^2}{(30)^4 + (15)^4} = 1.4706.$$

Interpolating between ψ = 1.4 and 1.5 in Table 3 yields a multiplying factor equal to 1.0941. The results are set out below for comparison.

$$\sigma_Y = 15 \text{ ft.}, \qquad \sigma_X = 30 \text{ ft.}$$

Exact ECEP = .8704 σ_X = 26.1120 ft., Reference [3]

Approx. ECEP = 1.1774(21.2132) = 24.9764 ft., Equation (12)

Approx. ECEP = 1.1774(22.5000) = 26.4915 ft., Equation (13)

Approx. ECEP = 1.1774(23.7171) = 27.9245 ft., Equation (14)

Approx. ECEP = 1.0941(23.7171) = 25.9489 ft., Equation (35)

The results confirm the remarks made in Section II about the relative worth of the first three approximations, i.e., only (13) has merit unless $c = \sigma_Y/\sigma_X$ is very close to one. The results also show that (35) is closer to the exact than (13) but not by much.

Let us now consider another example with a smaller c. Consider σ_Y = 15 and σ_X = 100 so that c = .15. In this case ψ = 1.0450 with corresponding multiplying factor .9714 obtained by interpolation.

$$\sigma_Y = 15 \text{ ft.}, \qquad \sigma_X = 100 \text{ ft.}$$

Exact ECEP = .6941 σ_X = 69.1600 ft., Reference [3]

Approx. ECEP = 1.1774(38.7298) = 45.6005 ft., Equation (12)

Approx. ECEP = 1.1774(57.5000) = 67.7005 ft., Equation (13)

Approx. ECEP = 1.1774(71.5017) = 84.1862 ft., Equation (14)

Approx. ECEP = .9714(71.5017) = 69.4568 ft., Equation (35) .

Here again we see equation (13) is closer to the exact than either (12) or (14) but barely close enough for practical use. On the other hand, approximation (35) is quite close to the exact, differing only in the third digit.

Thus far the comments in this section have been confined to the bivariate case. As aforementioned, the need for such an approximation is not great for the bivariate case since we have tables of the exact solution in Reference [3]. However, when the number of dimensions exceeds two, tables of the exact solution and/or computer programs

for the exact solution are not believed to exist. Hence, in the general multivariate case, an approximation for the radius of the 100P% k-sphere could be of practical value.

To obtain such an approximation, we apply the above limiting concept to the general expression for L in (34) vice the two dimensional expression in (24). This provides us with the following approximation to S_p, the radius of the 100P% k-sphere in the multivariate case:

$$S_p \approx \left\{ \frac{k\chi^2_{\nu,P}}{\nu} \right\}^{\frac{1}{2}} \left[(\sigma_1^2 + \ldots + \sigma_k^2)/k \right]^{\frac{1}{2}} \tag{36}$$

$$\nu = \frac{(\sigma_1^2 + \ldots + \sigma_k^2)^2}{\sigma_1^4 + \ldots + \sigma_k^4} \quad .$$

The worth of this approximation is difficult to ascertain due to the absence of tables of or programs for the exact solution. However, some rough comparisons can be made if we once more resort to simulation. To make these comparisons, we specify k, $\sigma_1, \ldots, \sigma_k$, and P and then compute our approximation to S_p which we shall designate S_p'. For the same σ_i, we then repeatedly sample from a multivariate normal population, each time sensing on the radial error R. The estimate of P, $\hat{P}$, is simply the ratio of the number of replicates in which R is less than S_p' to the total number of replicates. If the approximation has merit, P should lie within sampling variation of $\hat{P}$.

Several example cases are considered below. The reader who is seriously interested in using approximation (36) should engage in a more extensive evaluation. For each case, the number of replicates was set at 10,000. Using this number, one is 95% confident that the true population proportion encompassed by a k-sphere of radius S_p' lies within .01 of $\hat{P}$. Hence, values of $\hat{P}$ within .01 of P indicate that the approximation has merit. The cases considered and the results are shown in Table 4.

TABLE 4

Case	k	P	σ_1	σ_2	σ_3	σ_4	S_P'	$\hat{P}$
1	3	.50	1	1	1		1.5382	.5031
2	3	.90	1	1	1		2.5003	.9038
3	3	.50	1	2	4		3.6367	.5032
4	3	.90	1	2	4		7.1374	.9020
5	4	.50	1	2	4	8	7.3596	.4965
6	4	.90	1	2	4	8	14.3186	.9019

In the first two cases in Table 4, the distribution is spherical so the values of S_P' for these cases are exact. We note that P lies within sampling variation of $\hat{P}$ for both values of P as expected. In the last four cases, the distribution is ellipsoidal and as such, values of S_P' are approximations. In all four cases, P lies within sampling variation of $\hat{P}$. Because of the small number of cases examined, we cannot conclusively state that the approximation is "good" for all cases. However, these cases are evidence that the approximation has promise.

VII. CONCLUDING REMARKS

The addition of this report to References [7] and [8] provides one with the methodology to construct (P, γ) tolerance k-spheres for the uncorrelated multivariate normal distribution. The spheres can be constructed whether the sample size is finite or infinite (corresponds to the case where the parameters are known) and whether the standard deviations are the same for each direction or different. References [7] and [8] address the case where the standard deviations are the same for all directions and provide the exact formula for the radius of the k-sphere. This report addresses the case where they are different and provides an approximation for the radius of the k-sphere. This approximation provides little loss in accuracy for the cases examined even for small sample sizes. Collectively, these reports provide the weapons analyst with the tools to make a more meaningful assessment of the delivery accuracy of a weapon than has been possible in the past. However, one closing cautionary point should be mentioned. These reports assume that there is no bias in the weapon system, i.e., that the long run average or expected miss distance is zero in all directions. If this assumption of zero bias cannot be made, the procedures set forth in this report and in References [7] and [8] are not directly applicable.

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